Some uses of moduli spaces in particle and field theory

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Abstract

In this talk I shall try to give an elementary introduction to certain areas of mathematical physics where the idea of moduli space is used to help solve problems or to further our understanding. In the wide area of gauge theory, I shall mention instantons, monopoles and duality. Then, under the general heading of string theory, I shall indicate briefly the use of moduli space in conformal field theory and M-theory.

Invited talk at the EWM Workshop on Moduli Spaces, 2–3 July, Oxford; Proceedings edited by Frances Kirwan, Sylvie Paycha, Tsou Sheung Tsun, produced by Mathematical Institute, Oxford University.

1 Introduction

Physicists seldom define their terms. So although I know roughly what a moduli space is, and the sort of thing one does with it in physics, I was not really very sure of what exactly it is. So I asked Frances (Kirwan), just as the Balliol College (where participants were lodged) porters did when they also wanted to know what a moduli space was. I have always taken it to be some sort of useful parameter space, convenient in the sense that mathematicians have already worked out all its properties (at least in the classical cases). But Frances told me something much more significant—she describes it as a parameter space in the *nicest possible way.*

So in the next 55 minutes or so, I shall try to give you a rough picture of how physicists have made use of this nice concept of a parameter space. We should note, however, that it is far from a one-way traffic. Much of the tremendous progress in 4-manifold theory, and a large part of it is done here, came about by studying certain moduli spaces occurring in mathematical physics.

A few notes of warning, however, are in place. For a hard-nosed or pragmatic physicist, (A) spacetime X has 4 dimensions, 3 space and 1 time, with an indefinite metric. By an indefinite metric I mean that the quadratic form giving the metric is not positive definite, so that two distinct points in spacetime can be null-separated. In fact, distances along light-paths are always zero. For him (or her) also (B) spacetime is by and large like $\mathbb{R}^4$, that is, (i) flat, (ii) looking more or less the same in all directions, (iii) real, and (iv) more or less infinite in all its 4 directions and hence non-compact.

On the other hand, algebraic geometry is more about Riemannian manifolds and the best results are almost always obtained for the compact case. In order to make contact, the concept of spacetime has to be modified in several significant ways.

1. One considers definite metrics, a process known as *euclideanization.* Then many nice things happen. In particular, the *wave operator*

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

which is hyperbolic, becomes the 4-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

which is elliptic, and for elliptic operators there are all sorts of good results like the index theorems. Euclideanization is done in the following: Self-dual Yang–Mills theory, instantons, monopoles, Seiberg–Witten theory, strings, ...

2. Alternatively, one *complexifies* spacetime, and then the question of definite or indefinite metric disappears. In this case, one can use powerful complex

manifold techniques including twistor theory. This is also where supersymmetry comes in mathematically. Moreoever, by a change of point of view (see later), Riemann surfaces also play an important role. Complexification is done in superstrings, supersymmetric Yang–Mills theory, M-theory, ...

3. One also changes the topology of spacetime by *compactifying* some or all of its directions. In some cases, this is only a mild change, amounting to imposing certain decay properties at infinity (see later). In other cases, this gives rise to important symmetries of the theory. Compactification is done in instantons, superstrings, M-theory, ...

4. One either changes the number of spacetime dimensions or re-interprets some of them as other degrees of freedom. This dimensional change is done in strings, superstrings, monopoles, M-theory, ...

At first sight, these modifications look drastic. The hope is that they somehow reflect important properties of the real physical world, and that the nice results we have do not disappear on us once we know how to *undo* the modifications. Surprisingly, the (largely unknown) mathematics underlying real 4-dimensional spacetime looks at present quite intractable!

2 Yang–Mills theory (Gauge theory)

Unlike most of the other theories I shall mention, Yang–Mills theory is an experimentally 'proven' theory. In fact, it is generally believed, even by hard-nosed or pragmatic physicists, that Yang–Mills theory is the basis of *all* of particle physics. From the physics point of view, Yang–Mills theory is the correct framework to encode the invariance of particle theory under the action of a symmetry group—the gauge group G—at each spacetime point. For example, let $\psi(x)$ be the wave-function of a quantum particle. Then the physical system is invariant under the action of the group:

$$\psi(x) \mapsto \Lambda(x)\psi(x), \quad \Lambda(x) \in G.$$

This invariance is known as *gauge invariance*. Now the groups that are most relevant to particle physics are $U(1), SU(2), SU(3)$. However, we shall come across other groups as well. But for simplicity, we shall take $G = SU(2)$, unless otherwise stated.

There is an additional ingredient in many favoured gauge theories, namely *supersymmetry*. This is a symmetry relating two kinds of particles: bosons (e.g. a photon) with integral spin and fermions (e.g. an electron) with half-integral spin. Spin is a kind of internal angular momentum which is inherently quantum mechanical. Since bosons and fermions in general behave quite differently (e.g. they obey different statistics), this symmetry is not observed in nature. However,

one can imagine this symmetry holding for example at ultra-high energies. What makes this symmetry theoretically interesting is that many theories simplify and often become complex analytic with this extra symmetry, making much of the underlying mathematics accessible. Also the complex analyticity links such theories with most studies of moduli spaces.

Mathematically, Yang–Mills theory can be modelled (in the simplest case) by a principal bundle P (see Figure 1) together with a connection on it. I remind

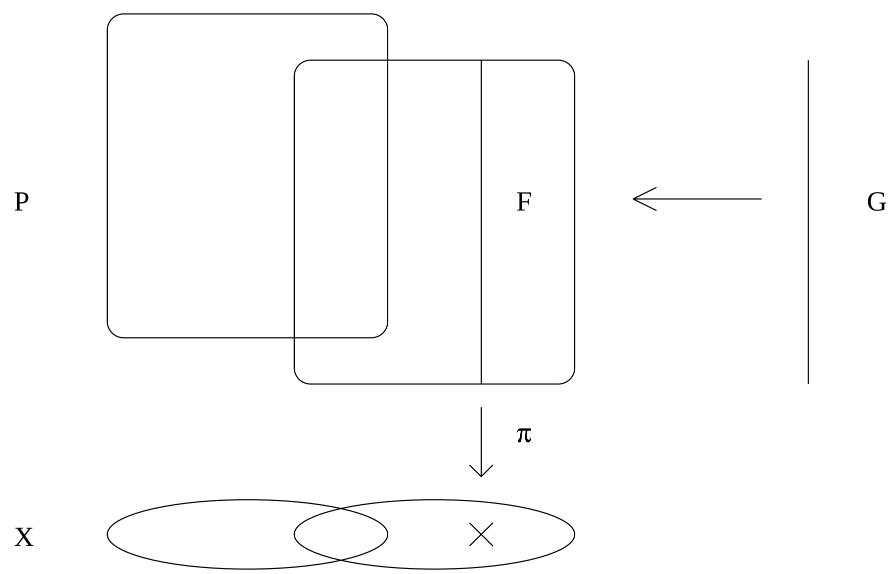

Figure 1: Sketch of a principal bundle

you that, roughly speaking, a *principal bundle* is a manifold P with a projection π onto a *base space* X, and a right action by the *structure group* G. In general, the base space can be any smooth manifold, but here we consider only the case of spacetime X. Above each point $x \in X$, the inverse image (called the *fibre*) $\pi^{-1}(x)$ is homeomorphic to G. The *total space* P is locally a product, in the sense that X is covered by open set U_α and $\pi^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times G$. A *connection* A is a 1-form on P with values in the Lie algebra $\mathfrak{g}$ of G, satisfying certain conditions and giving a prescription for differentiating vectors and tensors on X. It combines with the usual exterior derivative d to give the covariant exterior derivative d_A:

$$d_A = d + A$$

in such a way as to preserve gauge invariance.

Next we need the curvature 2-form:

$$F_A = dA + AA \quad (F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\mu, A_\nu]).$$

The second formula (in brackets) is the same as the first one, but written in local coordinates, or 'with indices', where $\mu = 0, 1, 2, 3$.

Since $\dim X = 4$ (for the moment, anyway), we have the Hodge star operator which takes 2-forms to 2-forms:

$$\begin{aligned} *\colon \quad \Omega^2 &\to \Omega^2 \\ F_A &\mapsto {}^*F_A. \end{aligned}$$

In local coordinates, this can be written as

$$^*F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma},$$

where $\epsilon_{\mu\nu\rho\sigma}$ is a completely skew symbol defined by $\epsilon_{0123} = 1$. Notice that

$$\begin{aligned} (*)^2 &= +1 \quad \text{in euclidean metric} \\ (*)^2 &= -1 \quad \text{in Minkowskian metric.} \end{aligned}$$

Yang–Mills theory is given by the Yang–Mills action or functional

$$S(A) = \frac{1}{8\pi^2}\int_X \mathrm{tr}(F_A{}^*F_A) = \frac{1}{8\pi^2}\|F_A\|^2.$$

The curvature satisfies:

$$\begin{aligned} d_A F_A &= 0 \quad \text{(Bianchi identity)} \\ d_A{}^*F_A &= 0 \quad \text{(Yang}-\text{Mills equation).} \end{aligned}$$

These are the *classical* equations for Yang–Mills theory. Notice that the first one is an identity from differential geometry, and the second one comes from the first variation of the action.

The space of connections $\mathcal{A}$ is an affine space, but we are really interested in connections modulo gauge equivalence. Two connections A, A' are gauge equivalent if they are 'gauge transforms' of each other:

$$A' = \Lambda^{-1}A\Lambda + \Lambda^{-1}d\Lambda.$$

In other words, $\Lambda(x) \in G$, Λ is a fibre-preserving automorphism of P invariant under the action of G. We shall use the symbol $\mathcal{G}$ for the group of gauge transformations Λ.

So we come to our first, most basic, *moduli space*

$$\bar{\mathcal{M}} = \mathcal{A}/\mathcal{G}.$$

It is in general infinite-dimensional with complicated topology.

We shall be interested in various subspaces or refinements of $\bar{\mathcal{M}}$.

One theoretical use of $\bar{\mathcal{M}}$ itself is in (the euclidean formulation of) quantum field theory, where with the Feynman path integral approach, one has to consider the integral of the exponential of the Yang–Mills action over $\bar{\mathcal{M}}$:

$$\int_{\bar{\mathcal{M}}} e^{-S(A)}.$$

But this integral is very difficult to define in general!

The moduli space $\bar{\mathcal{M}}$ has a singular set which represents the *reducible connections*, which are connections with holonomy group $H \subset G$ such that the centralizer of H properly contains the centre of G. We say then that the connection *reduces* to H. The complement $\mathcal{M}$ of this singular set is dense in $\bar{\mathcal{M}}$, and represents the irreducible connections. For $G = SU(2)$, near an irreducible connection $\bar{\mathcal{M}}$ is smooth, but reducible connections lead to cone-like singularities in $\bar{\mathcal{M}}$.

2.1 Instantons

Recall that $G = SU(2)$. Bundles P over X are classified by the second Chern class of the associated rank 2 vector bundle E (cf. Rosa-Maria Miró-Roig's talk):

$$k = c_2(E)[X] = \frac{1}{8\pi^2}\int_X \operatorname{tr} F_A^2 \in \mathbb{Z}.$$

We say that a connection A is self-dual (or anti-self-dual) if its curvature F_A satisfies

$$F_A = {}^*F_A \quad (\text{resp. } F_A = -{}^*F_A).$$

Then given any connection A, we can decompose the corresponding curvature F_A into its self-dual and and anti-self-dual parts:

$$F_A = F_A^+ + F_A^-.$$

In the context of Yang–Mills theory a self-dual connection is called an *instanton*[1]:

$$F_A = {}^*F_A \Leftrightarrow F_A^- = 0.$$

In this case,

$$\text{Bianchi identity} \cong \text{Yang}-\text{Mills equation}.$$

In other words, a self-dual connection is *automatically* a classical solution.

Now we have

$$\begin{aligned} S(A) &= \frac{1}{8\pi^2}\int_X |F_A^+|^2 + |F_A^-|^2 \\ k &= \frac{1}{8\pi^2}\int_X |F_A^+|^2 - |F_A^-|^2. \end{aligned}$$

[1]It is a matter of convention whether one so defines a self-dual or anti-self-dual connection.

Hence one has immediately
$$S(A) \geq k,$$
and
$$S(A) = k \Leftrightarrow F_A^- = 0.$$
So a self-dual connection gives an absolute minimum for the action. The integer k is known as the *instanton number*.

Warning: Nontrivial self-dual connections exist only when X is either euclidean or complex.

The *mathematical* magic of instantons is that instead of solving the second order Yang–Mills equations we have only the first order self-duality equation to deal with. These connections can actually be constructed using euclidean *twistor methods* without explicitly solving any equations (cf. Tatiana Ivanova's talk).

Physically, the presence of instanton contribution in the path integral allows tunnelling between different vacua (i.e. lowest energy states) of the relevant Yang–Mills theory (namely quantum chromodynamics for strong interactions or QCD). This role of the instantons can be compared to lower-dimensional objects such as 'solitons' or topological defects called 'kinks' which connect up two different states at infinity (see Figure 2). The two phenomena are quite similar, since 'tunnelling' means a quantum particle can penetrate a potential barrier which a classical particle cannot go through, thus connecting two classically separate states. The

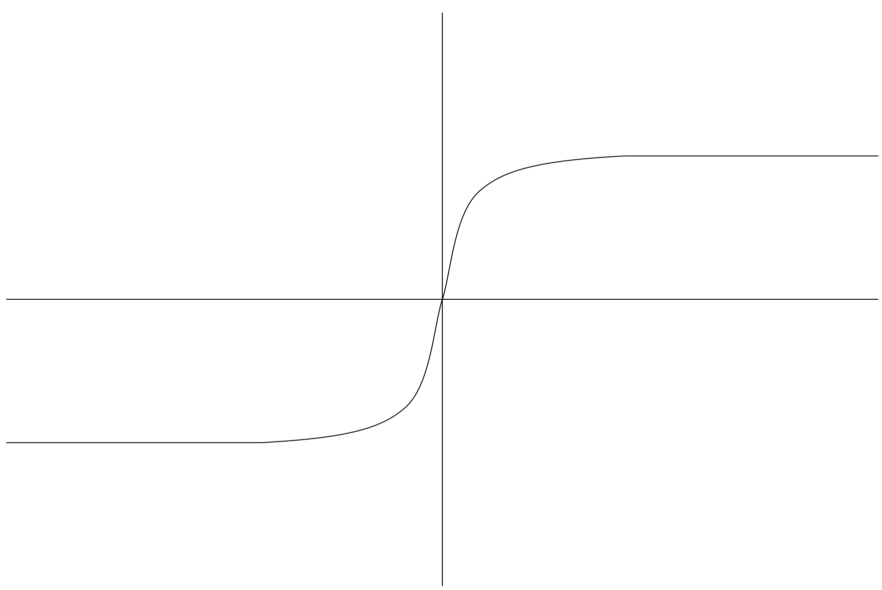

Figure 2: Sketch of a kink connecting two different states

effect of instantons is 'non-perturbative' in the sense that such an effect cannot be obtained as a term in a power series expansion of g the coupling constant (which is measure of the 'strength' of the interaction under consideration, and

which appears for example in the nonlinear term of the curvature form $F_{\mu\nu}$). This is a direct manifestation of the fact that instantons are topological in nature and cannot be obtained by any 'local' considerations such as power series expansions.

Since in euclidean space the Yang–Mills equations are elliptic, and concentrating on irreducible connections gets rid of zero eigenvalues, one can use the index theorem to count the 'formal dimension' of *instanton moduli space*. Typically the smooth part of the moduli space will have this formal dimension as its actual dimension. For example,

$$X = S^4, \quad \dim_{\mathbb{C}}(\mathcal{M}_{I,k}) = 8k - 3.$$

Uhlenbeck has given a unique compactification of $\mathcal{M}_I$, the union for all k. For more details about instanton moduli spaces, I again refer you to Tatiana Ivanova's talk.

2.2 Monopoles

Recall $G = SU(2)$.

Consider a Yang–Mills theory with a scalar field (called *Higgs field*) ϕ, together with a potential term $V(\phi)$ which is added to the Yang–Mills action. Suppose further that

$$V(\phi_0) = \text{minimum for } |\phi_0| \neq 0,$$

and that $V(\phi)$ is invariant under a subgroup $U(1) \subset SU(2)$. Then for those connections of P which are reducible to this $U(1)$ subgroup, we can for certain purposes concentrate on this 'residual gauge symmetry' and have a $U(1)$ gauge theory. If we interpret this $U(1)$ as Maxwell's theory of electromagnetism, then a non-trivial reduction of P can be regarded as a *magnetic monopole*. The *magnetic charge* k is given by the first Chern class of the reduced bundle. In fact we have the following exact sequence which gives us an isomorphism:

$$\begin{array}{ccccccc} \pi_2(SU(2)) & \rightarrow & \pi_2(SU(2)/U(1)) & \xrightarrow{\sim} & \pi_1(U(1)) & \rightarrow & \pi_1(SU(2)) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

Unlike the original magnetic monopole considered by Dirac, these 't Hooft–Polyakov monopoles have finite energy and are the soliton solutions of the field equations corresponding to the action:

$$S(A, \phi) = S(A) + \|D\phi\|^2 + \lambda(1 - |\phi|^2)^2,$$

where the last term is the usual form of the potential $V(\phi)$. From this we get the Yang–Mills–Higgs equations (YMH):

$$\begin{aligned} D_A F &= 0, \\ {D_A}^* F &= -[\phi, D_A \phi], \\ {D_A}^* D_A \phi &= 2\lambda\phi(|\phi|^2 - 1). \end{aligned}$$

Now we specialize to a certain limit, the Prasad-Somerfeld limit: $V(\phi) = 0$, but $|\phi| \to 1$ at infinity. Then the Yang–Mills–Higgs system becomes:

$$\begin{aligned} D_A F &= 0, \\ {D_A}^* F &= -[\phi, D_A\phi], \\ {D_A}^* D_A \phi &= 0. \end{aligned}$$

Consider next a Yang–Mills theory in euclidean $\mathbb{R}^4$, invariant under x_4-translations. Then we can write

$$A = A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + \phi dx_4,$$

where A_1, A_2, A_3, ϕ are Lie algebra-valued functions on $\mathbb{R}^3$. The action can be written as

$$S(A) = \|F_A\|^2 = \|F\|^2 + \|D\phi\|^2,$$

where now F is the curvature of the connections in 3 dimensions:

$$A' = A_1 dx_1 + A_2 dx_2 + A_3 dx_3,$$

and D is the corresponding 3-dimensional covariant derivative. In this way, we can make the following identification since the actions for the two theories are identical:

$$\text{YMH on } \mathbb{R}^3 \cong \text{dimensionally reduced YM on } \mathbb{R}^4.$$

In this case,

$$F_A = {}^*F_A \Rightarrow \text{first 2 YMH}.$$

Hence a solution to the Bogomolny equation

$$F = {}^*D_A\phi$$

gives a solution of YMH. These are known as 'static monopoles'.

The moduli spaces $\mathcal{M}_k$ corresponding to a given charge k are well studied, at least for $k = 1, 2$. The translation group $\mathbb{R}^3$ acts freely on $\mathcal{M}_k$, so does an overall phase factor S^1. Dividing these out we get the reduced monopole moduli spaces $\mathcal{M}_k^0$, $\dim_{\mathbb{C}} = 4k - 4$. Taking the k-fold covers, one obtains:

$$\tilde{\mathcal{M}}_k \cong \mathbb{R}^3 \times S^1 \times \tilde{\mathcal{M}}_k^0.$$

The special case of $k = 2$ has been studied by Atiyah and Hitchin as an entirely novel way of obtaining the scattering properties of two monopoles, using a metric on $\mathcal{M}_2^0$ they discovered, and assuming (with Manton) that geodesic motion on it describes adiabatic motion of the two monopoles. This is the most direct use that I know of of moduli space for deriving something akin to dynamics!

2.3 Topological field theory

I wish just to mention a class of quantum field theories called *topological quantum field theories* (TQFT), where the observables (correlation functions) depend only on the global features of the space on which these theories are defined, and are independent of the metric (which, however, may appear in the classical theory). Atiyah gave an axiomatic approach to these, but there are so many local experts here that I do not feel justified in expanding on that!

Instead, I shall just indicate the role of moduli space in Witten's approach. Starting with a moduli space $\mathcal{M}$ one can get fields, equations and symmetries of the theory. Witten postulates the existence of certain operators $\mathcal{O}_i$ corresponding to cohomology classes η_i of $\mathcal{M}$ such that

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \int_{\mathcal{M}} \eta_1 \cdots \eta_n,$$

where $\langle \cdots \rangle$ denotes the correlation function of the operators. Hence he obtains these correlation functions as intersection numbers of $\mathcal{M}$, using Donaldson theory. So in a sense the TQFT is entirely defined by $\mathcal{M}$.

The observables called correlation functions can best be understood in the case of, for example, a 2-point function in statistical mechanics. This is the probability, given particle 1, of finding particle 2 at another fixed location.

To go into any further details about TQFT would require more detailed knowledge both of quantum field theory and supersymmetry. These would lead us unfortunately too far from the context of this workshop.

2.4 Seiberg–Witten theory

Recall that a spin structure on X is a lift of the structure group of the tangent bundle of X from $SO(4)$ to its double cover $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$. Because of this isomorphism, one can represent a spin structure more concretely as a pair of complex 2-plane bundles $S^+, S^- \to X$, each with structure group $SU(2)$. A slightly more general concept is a spin^c structure over X, which is given by a pair of vector bundles W^+, W^- over X with an isomorphism for the second exterior powers

$$\Lambda^2 W^+ = \Lambda^2 W^- = L, \text{ say,}$$

such that one has locally

$$W^\pm = S^1 \otimes L^{\frac{1}{2}},$$

where $L^{\frac{1}{2}}$ is a local square root of L: $L^{\frac{1}{2}} \otimes L^{\frac{1}{2}} = L$.

Given a spin^c manifold X, the Seiberg–Witten equations (SW) are written for a system consisting of 1) a unitary connection A on $L = \Lambda^2 W^\pm$, and 2) ψ a section of W^+. Then these equations are:

$$\begin{aligned} D_A \psi &= 0 \\ F_A^+ &= -\tau(\psi, \psi), \end{aligned}$$

where τ is a sesquilinear map $\tau\colon W^+ \times W^+ \to \Lambda^+ \otimes \mathbb{C}$.

The Seiberg–Witten equations (SW) can be obtained from varying the following functional:

$$E(A,\psi) = \int_X |D_A\psi|^2 + |F_A^+ + \tau(\psi,\psi)|^2 + R^2/8 + 2\pi^2 c_1(L)^2,$$

where R is the scalar curvature of X and $c_1(L)$ is the first Chern class of L. Notice that the last two terms depend only on X and L, so that solutions of SW are absolute minima of E on the given bundle L.

The relevant moduli space here is the space $\mathcal{M}$ of all irreducible solution pairs (A,ψ), modulo gauge transformations. The Seiberg–Witten invariants are then homology classes of $\mathcal{M}$, independent of the metric on X. These invariants prove very useful in 4-manifold theory. In particular, Seiberg and Witten give a 'physicist's proof' that the instanton invariants of certain 4-manifolds (namely with $b^+ > 1$, where b^+ is the dimension of the space of self-dual harmonic forms) can be expressed in terms of the Seiberg–Witten invariants.

From the quantum field theory point of view, the importance of Seiberg–Witten theory lies in the concept of *duality*. In a modified version of Yang–Mills theory, called $N = 2$ supersymmetric Yang–Mills theory, the quantum field theory is described by a scale parameter t and a complex parameter u (here supersymmetry is essential). In the limit $t \to \infty$, the theory is described by an analytic function τ of u. If $b^+(X) > 1$, then τ is *modular* (in the classical sense) with respect to the action of $SL(2,\mathbb{Z})$. This means in particular that a theory with parameter u is related to a theory with parameter u^{-1} in a definite and known way. The transformation $u \mapsto u^{-1}$ corresponds to changing the coupling constant to its inverse. Hence for the magnetic monopoles of the theory this represents a *duality* transformation: from *electric* with coupling e to *magnetic* with coupling $\tilde{e}$ and vice versa, since Dirac's quantization condition states that $e\tilde{e} = 1$ in suitable units. By relating a 'strongly coupled' theory to a 'weakly coupled' theory, one can hope to obtain results on the former by performing perturbative calculations (which are meaningless when coupling is strong) in the latter. By inspecting their moduli spaces one is often able to identify pairs of dually related theories.

3 String and related theories

I shall be extremely brief about these theories. The reason is, apart from my own obvious ignorance, that they are considerably more complicated than gauge theories and require much more knowledge not only of quantum physics but also of algebraic geometry than can reasonably be dealt with in this workshop. My aim here is just to give a taste of some immensely active areas of research in

mathematical physics in recent years where moduli spaces play an important role.

The gist of string theory is that the fundamental objects under study are not point-like particles as in gauge field theories but 1-dimensional extended strings. These strings are really the microscopic quantum analogues of violin strings: they move in space and they also vibrate. The equation of motion of a free string can be obtained from an action which is similar to that for a massless free particle. In the latter case we have

$$S_0 = \int d\tau \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

which is just the *length* of the 'worldline' in spacetime X traced out by the particle as it travels through space. Here $\eta_{\mu\nu}$ is the metric on X and x^μ are the coordinates of the particle. For the string the free action is the *area* of the 'worldsheet' (with coordinates σ, τ) traced out by the 1-dimensional string in spacetime X:

$$S_1 = \int d\sigma d\tau \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu,$$

where the indices $\alpha, \beta = 0, 1$ refer to the worldsheet. Varying S_1 with respect to x gives simply the 2-dimensional wave equation:

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) x^\mu = 0.$$

We see that in this context spacetime coordinates can be regarded as fields on the 2-dimensional surface which is the worldsheet.

Interaction between strings are given by the joining and splitting of strings so that the resultant worldsheet can be visualized, on euclideanization, as a Riemann surface Σ with a given genus (see Figure 3). For example, a hole in Σ can be obtained by one closed string splitting into two and then joining together again. In fact, a useful way of looking at string theory is to think of it as being given by an embedding f of a Riemann surface Σ into spacetime X (Figure 4).

3.1 Conformal field theory

We have written the action S_1 for a free string in terms of a particular parametrization of Σ, but obviously the physics ought to be invariant under reparametrization. The group of reparametrization on Σ is the infinite-dimensional conformal group, and that is the symmetry group of string theory.

On the other hand, on a given Riemann surface Σ one can consider certain field theories which have this invariance. These are called conformal field theories (CFT) and play important roles in statistical mechanics and critical phenomena (e.g. phase change), when the theories become independent of the length scale (so that quantities are defined only up to conformal transformations).

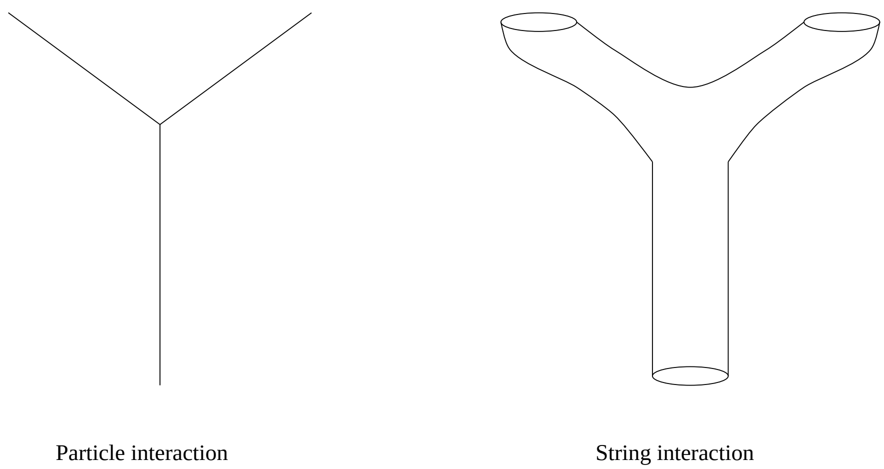

Figure 3: Schematic representation of particle and string interactions

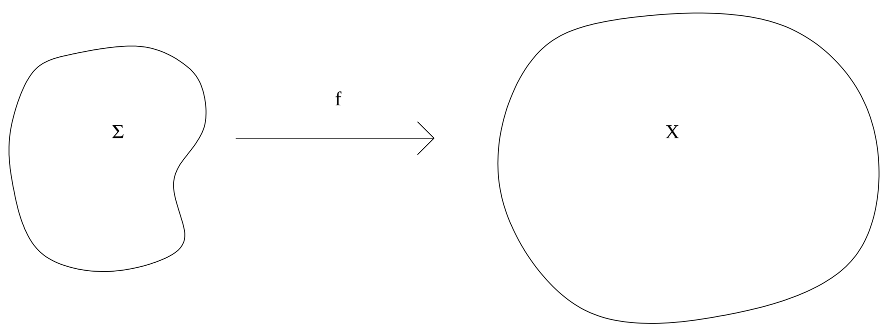

Figure 4: Embedding worldsheet into spacetime

The concept of moduli plays an important role in CFT. In fact, the original idea of modulus is defined for Riemann surfaces (see talk by Frances Kirwan). So a torus T^2 has one modulus τ (see Figure 5). The conformal structure of T^2 is invariant under the action of the *modular group* $SL(2,\mathbb{Z})$ on τ.

CFT are often studied for their own sake, but as far as string theories are concerned their use lies in the fact that they are the terms in a first-quantized, perturbative formulation of string theory. Schematically, one can think of string theory as the 'sum over g' of CFT on Riemann surfaces of genus g. Unfortunately, this 'summation' has never yet been given a precise meaning. What provides some hope that the problem may be tractable is the fact that the infinite-dimensional integral $\int e^{-S_1(x)}$ occurring in the path integral formalism can be reduced to one on the moduli space of the Riemann surface, which is finite-dimensional.

3.2 Various string theories

Up to now I have been carefully vague about the nature of spacetime X in string theory. It turns out that to get a consistent, first-quantized theory, one needs X to

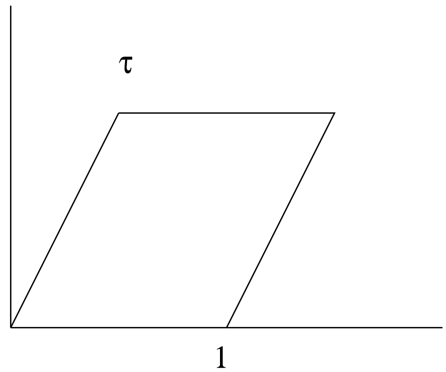

Figure 5: A 2-torus represented on the complex plane

have 26 dimensions! If we modify the theory by adding supersymmetry to produce a *superstring* theory, then $\dim X = 10$. However, this potentially disastrous requirement has been turned to good use to produce interesting theories in 4 dimensions, as we now briefly sketch.

We shall concentrate on the supersymmetric version as being the more favoured by string theorists, in that we now assume $\dim X = 10$. Imagine that one can compactify 6 of these 10 dimensions so that

$$X \cong K \times \mathbb{R}^4$$

with K a compact 6-dimensional space, and moreover that the size of K is small. Since the length is an inverse measure of energy, this means that to observers of low energy (such as us) spacetime will just look 4-dimensional and the other 6 dimensions are curled up so tight we cannot see them. The often-quoted example is that a water pipe looks like a thin line from a distance.

Not only that, the symmetries of X can be factored into that of $\mathbb{R}^4$ (the usual ones) and that of K. The latter can then be interpreted as the internal symmetries of Yang–Mills theory. In fact, the choice of K is dictated by which gauge symmetry one wants.

There are in all 5 string theories. A string can be open (homeomorphic to an interval) or closed (homeomorphic to a circle). An open string theory is called Type I. For closed strings, depending in the boundary conditions one imposes, one has Type IIA or Type IIB. If one combines both the usual and the supersymmetric versions one obtains the heterotic string, with gauge group (after suitable compactification) either $E_8 \times E_8$ or $SO(32)$. The $E_8 \times E_8$ heterotic string is particularly favoured as being able to include various Yang–Mills theories which are important in particle physics.

3.3 M Theory

One can generalize the 1-dimensional strings to higher-dimensional objects called 'membranes'; similarly superstrings to 'supermembranes'. The study of these last objects have become particularly fashionable, especially after the introduction of something called M-theory.

Now supersymmetry can also be made into a local gauge theory which is then called *supergravity*. It was shown some time ago that in supergravity, $\dim X \leq 11$, so 11-dimensional supergravity was studied as being in some sense a unique theory.

M-theory is perceived as an 11-dimensional supergravity theory, where the 11-dimensional manifold X can be variously compactified to give different superstring theories. Moreover, solitonic solutions are found which are supermembranes. By examining the moduli of these solutions one can connect pairs of underlying string theories. For example, reminiscent of the Seiberg–Witten duality and using the modular transformations on the modulus τ of the torus (in one of the compactifications of X), one can connect the two different versions of the heterotic string. In fact, by using both compactification and duality one finds that M-theory can give rise to all the 5 superstring theories mentioned above. So in some sense, all the 5 are equivalent and one can imagine that they are just different perturbative expansions of the same underlying M-theory.

Most recently, Maldecena suggested that M-theory on compactification on a particular 5-dimensional manifold (called anti-de Sitter space), including all its gravitational interactions, may be described by a (non-gravitational) Yang–Mills theory on the boundary of X which happens to be 4-dimensional Minkowski space (i.e. flat spacetime). This opens up some new vistas in the field.

Although progress is made in an almost day-to-day basis, we are still waiting for a fuller description, perhaps even a definition, of M-theory. Meanwhile, it has generated a lot of interest and especially intense study into the various moduli spaces that occur.

4 Conclusions

I have endeavoured to describe a few pysical theories in which moduli space plays an important role. However, I must say that the success in the reverse direction is more spectacular—using Yang–Mills moduli spaces (in different specializations) to understand 4-manifolds, following Donaldson, Kronheimer and many others. At the beginning I have explained why the success in physics is more restricted. Nevertheless, there are many high points:

1. Self-dual Yang–Mills $\rightsquigarrow$ instantons $\rightsquigarrow$ vacuum structure of QCD.

2. Monopole moduli spaces $\rightsquigarrow$ identification of pairs of dual theories in Seiberg–Witten scheme $\rightsquigarrow$ hope for possibility of practical computations in quantum field theory.

3. Classification of conformal field theories $\rightsquigarrow$ application of theoretical statistical mechanics.

4. Identifying moduli spaces to connect up the different string theories $\rightsquigarrow$ leading to a unification in 11 dimensions?

But for lack of time and expertise, I have omitted many other areas of mathematical physics being actively pursued at present in which moduli spaces play significant roles.

References

The following is only a small selection of articles that I have used in preparing this talk. They are in no way even representative.

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